Spectrum and generation of solutions of the Toda lattice

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Abstract

Given the tridiagonal matrix $J(t)$ defining a Toda lattice solution, the dynamic behavior of zeros of polynomials associated to $J(t)$ is analyzed. Also, under certain conditions the invariance of the spectrum of $J(t)$ is established. Finally, an example of solution is presented, and the method given in [2] to obtain new solutions is illustrated.

1 Introduction

We consider the following Toda lattice,

$$\left.\begin{array}{rcl} \dot{\alpha}_n(t) & = & \lambda_{n+1}^2(t) - \lambda_n^2(t) \\ \dot{\lambda}_{n+1}(t) & = & \frac{1}{2}\lambda_{n+1}(t)\left(\alpha_{n+1}(t) - \alpha_n(t)\right) \end{array}\right\}, \quad n \in \mathbb{N} \quad (\lambda_1 \equiv 0)\,, \tag{1}$$

where $\lambda_n(t)\,,\, \alpha_n(t)$ are complex and differentiable functions of one real variable such that $\dot{\alpha}_i\,,\, \dot{\lambda}_i$ denote the derivatives of $\lambda_n(t) \neq 0\,,\, t \in \mathbb{R}\,,\, n \geq 2\,,$. It is well known (see [7, pag. 705]) that (1) can be expressed in Lax pair form, like

$$\dot{J}(t) = [J(t), K(t)]\,, \tag{2}$$

being $[A, B] = AB - BA$ the commutator of operators A and B, and being $J(t),\, , K(t)$ the operators which matricial representation is given, respectively, by

$$J(t) = \begin{pmatrix} \alpha_1(t) & \lambda_2(t) & & \\ \lambda_2(t) & \alpha_2(t) & \lambda_3(t) & \\ & \lambda_3(t) & \alpha_3(t) & \ddots \\ & & \ddots & \ddots \end{pmatrix}, \quad K(t) = \frac{1}{2}\begin{pmatrix} 0 & -\lambda_2(t) & & \\ \lambda_2(t) & 0 & -\lambda_3(t) & \\ & \lambda_3(t) & 0 & \ddots \\ & & \ddots & \ddots \end{pmatrix} \tag{3}$$

with respect to the canonical base $\{e_i\}\,,\, i \geq 0\,$. (In the following, we denote in the same way each operator and its matricial representation with respect to this base.)

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The system (1) is a articular case of the generalized Toda lattice of order p,

$$\left.\begin{array}{rcl} \dot{J}_{nn}(t) & = & J_{n,n+1}(t)J^p_{n,n+1}(t) - J_{n-1,n}(t)J^p_{n-1,n}(t) \\ \dot{J}_{n,n+1}(t) & = & \frac{1}{2}J_{n,n+1}(t)\left[J^p_{n+1,n+1}(t) - J^p_{n,n}(t)\right] \end{array}\right\}, \quad n = 0, 1, \ldots, \tag{4}$$

where we denote by $J_n(t)$ the finite-dimensional section of order n of $J(t)$, defined by the first n rows and columns of $J(t)$ (see [3]). The generalized lattice (4) was proposed in [1] and, latter, was studied in [2]. There, an important tool in the study of solutions of (4) was the sequence $\{P_n(t,z)\}$ of polynomials defined by the three-term recurrence relation

$$\left.\begin{array}{l} P_{n+1}(t,z) = (z - \alpha_{n+1}(t))P_n(t,z) - \lambda^2_{n+1}(t)P_{n-1}(t,z), \quad n \geq 0 \\ P_{-1}(t,z) \equiv 0, \quad P_0(t,z) \equiv 1 \end{array}\right\}. \tag{5}$$

From $\{P_n(t,z)\}$ we can define the sequence $\{\widehat{p}_n(t,z)\}$ by

$$\widehat{p}_n(t,z) = \frac{P_n(t,z)}{\lambda_2(t)\cdots\lambda_{n+1}(t)}, \quad n \in \mathbb{N}.$$

Obviously, the zeros of $\widehat{p}_n(t,z)$ and $P_n(t,z)$ are the same.

Beside some other results, the basis of a method for obtaining new solutions of (4) from a given solution were established in [2]. In that paper, the existence of a point $C \in \mathbb{C}$ which is not a root of any polynomial $\widehat{p}_n(t,z)$ was determined as a necessary condition in the construction of such method. From (5) it can be easily established that

$$P_n(t,z) = \det(zI_n - J_n(t))$$

(see, for instance, [6, Ex. 5.7, pag. 30]). In other words, for any $t \in \mathbb{R}$ and $n \in \mathbb{N}$ the set of zeros of $P_n(t,z)$ coincides with the spectrum $\sigma(J_n(t))$ of $J_n(t)$. Therefore, if we want to find a point $C \in \mathbb{C}$ such that $P_n(t,z) \neq 0$ for any $n \in \mathbb{N}$, then we need to know some bound for the set of zeros of $P_n(t,z)$, $n \in \mathbb{N}$. This fact shows the relevance of knowing the dynamic behavior of such zeros, established in the following result.

Theorem 1 *Let $z_{n1}(t), z_{n2}(t), \ldots, z_{nn}(t)$ be the roots of $P_n(t,z)$, non necessary distinct. Then we have*

$$\dot{z}_{nk}(t) = \frac{\left(\widehat{p}_{n-1}(t, z_{nk}(t))\right)^2}{\sum_{j=0}^{n-1}\left(\widehat{p}_j(t, z_{nk}(t))\right)^2}, \tag{6}$$

understanding $\dot{z}_{nk}(t) = \infty$ when the multiplicity of $z_{nk}(t)$ as a zero of $P_n(t,z)$ would be $m(z_{nk}(t)) > 1$.

On the other hand, the relationship between the spectrum $\sigma(A)$ of a banded infinite matrix A and the spectrum $\sigma(A_n)$ of its main sections was analyzed, under certain conditions, in [4]. Concretely, for this kind of operators the representation $A = \Re A + i\Im A$ was used, assuming $\Re A$ selfadjoint and $\Im A$ bounded. In our case, if we assume that $J(t)$ verifies this restriction, then we have

$$J(t) = \Re J(t) + i\Im J(t), \quad t \in \mathbb{R},$$

where $\Re J(t)$ is a selfadjoint operator and $\Im J(t)$ is bounded. For $C \in \mathbb{R}$ verifying

$$d\left(C, \sigma\left(\Re J_n(t)\right)\right) > \|\Im J_n(t)\|,$$

from [4, Lemma 1, Lemma 2] we know $P_n(t,C) \neq 0$ or, what is the same, $C \notin \sigma(J_n(t))$. Moreover, taking into account that $\|\Im J(t)\| \geq \|\Im J_n(t)\|$, from these results we can deduce $P_n(t,C) \neq 0$ for any $n \in \mathbb{N}$ when $C \in \mathbb{C}$ is such that$d(C, \sigma(\Re J(t))) > \|\Im J(t)\|$. In this way, the zeros of each sequence of polynomials $\{P_n(t,z)\}$, $n \in \mathbb{N}$, are located in the neighborhood of $\sigma(\Re J(t))$ given by

$$\{z : d(z, \sigma(\Re J(t))) \leq \|\Im J(t)\|\}.$$

This fact justify the interest in obtaining relationship between $\sigma(J(t))$ for diferent values of $t \in \mathbb{R}$, because the bounding of zeros in a certain region of the complex plane permits to work with the method given in [2] in the complementary and free of zeros region. Moreover, in the case $\Im J(t) = 0$, this is, when $J(t)$ is a selfadjoint operator, it is well know the fact that $\sigma(J(t))$ is invariant on $t \in \mathbb{R}$ ([11]). In this case, due to some properties of the real Toda lattice (see for instance [7]), the associated Cauchy problem can be solved, recovering the solution $J(t)$ from the initial values defined by $J(0)$. We would like to establish some similar result when $\Im J(t) \neq 0$ in the more general possible situation. However, we think that some advance, in this sense, is a relevant contribution in the study of solutions of the Toda lattice. This fact is related with our next result.

Theorem 2 *Let $\{\alpha_n(t), \lambda_{n+1}(t)\}$, $n \in \mathbb{N}$ be a solution of (1) such that the sequence $\{\lambda_{n+1}(t)\}$, $n \in \mathbb{N}$, is bounded for each $t \in \mathbb{R}$. Then we have*

$$\sigma(J(t)) = \sigma(J(t_0)), \quad \forall t, t_0 \in \mathbb{R},$$

this is, the spectrum of $J(t)$ is invariant on $t \in \mathbb{R}$.

Second section is devoted to prove Theorems 1 and 2. In second 3, the construction of a solution of (1) from another solution will be show. Before, the existence of $C \in \mathbb{C}$ such that $P_n(t,C) \neq 0$ for any $n \in \mathbb{N}, t \in \mathbb{R}$, will be guaranteed

2 Invariance of spectrum vs. variation of zeros of polynomials

2.1 Proof of Theorem 1

Taking $p = 1$ in (10) of [2, Th. 2] we obtain

$$\dot{P}_n(t,z) = -\lambda_{n+1}^2(t) P_{n-1}(t,z) \tag{7}$$

for each $n \in \mathbb{N}$ and all $z \in \mathbb{C}$. Then, writing

$$P_n(t,z) = \prod_{i=1}^{n} (z - z_{ni}(t))$$

and taking derivatives with respect to t, we have

$$\dot{P}_n(t,z) = -\sum_{i=1}^{n} \dot{z}_{ni}(t) \prod_{j \neq i} (z - z_{ni}(t)). \tag{8}$$

With the notation established in Section 1, for each fixed zero $z = z_{nk}(t)$ of $P_n(t, z)$ the right hand side of (7) is not zero. In fact, we have $\lambda_{n+1}(t) \neq 0$ and, if we suppose $P_n(t, z_{nk}(t)) = P_{n-1}(t, z_{nk}(t)) = 0$, then using the recurrence relation (5) we will arrive to $P_{n-2}(t, z_{nk}(t)) = 0$ and, iterating, to $P_0(t, z_{nk}(t)) = 0$, which is not possible being $P_0 \equiv 1$.

Comparing (7) and (8) for $z = z_{nk}(t)$ we see

$$\sum_{i=1}^{n} \dot{z}_{ni}(t) \prod_{j \neq i} (z_{nk}(t) - z_{nj}(t)) = \lambda_{n+1}^2(t) P_{n-1}(t, z_{nk}(t)), \quad k = 1, \ldots, n. \tag{9}$$

Moreover, $\prod_{j \neq i}(z_{nk}(t) - z_{nj}(t)) = 0$ when $i \neq k$. Therefore, from (9) we have

$$\dot{z}_{nk}(t) \prod_{j \neq k} (z_{nk}(t) - z_{nj}(t)) = \lambda_{n+1}^2(t) P_{n-1}(t, z_{nk}(t)), \quad k = 1, \ldots, n, \tag{10}$$

and, consequently,

$$\dot{z}_{nk}(t) \prod_{j \neq k} (z_{nk}(t) - z_{nj}(t)) \neq 0, \quad k = 1, \ldots, n. \tag{11}$$

We shall take in consideration the two possible cases:

i) If the multiplicity of $z_{nk}(t)$ as a zero of $P_n(t, z)$ is $m_{nk}(t) > 1$, then the factor $z_{nk}(t) - z_{nk}(t)$ is in the left hand side of (11), so $\dot{z}_{nk}(t) = \infty$.

ii) If $z_{nk}(t)$ is a simple zero of $P_n(t, z)$, then from (10) we obtain

$$\dot{z}_{nk}(t) = \frac{\lambda_{n+1}^2(t) P_{n-1}(t, z_{nk}(t))}{\prod_{j \neq k}(z_{nk}(t) - z_{nj}(t))}. \tag{12}$$

On the other hand, writing

$$P_n(t, z(t)) = \prod_{i=1}^{n} (z - z_{ni}(t))$$

and, taking derivatives with respect to z,

$$P_n'(t, z) = \sum_{i=1}^{n} \prod_{j \neq i} (z - z_{nj}(t)).$$

So,

$$P_n'(t, z_{nk}(t)) = \prod_{j \neq k} (z_{nk}(t) - z_{nj}(t)). \tag{13}$$

Moreover, it is well known the expression

$$\sum_{j=0}^{n-1} (\widehat{p}_j(t, z_{nk}(t)))^2 = \frac{P_n'(t, z_{nk}(t)) P_{n-1}(t, z_{nk}(t))}{(\lambda_2(t) \cdots \lambda_{n+1}(t))^2} \tag{14}$$

(see [6, pag. 24]).

Finally, from (12), (13) and (14) we arrive to (6).

We remark that (14) holds in the case i), when $z_{nk}(t)$ is not a simple zero and the denominator in (6) is zero.

$\square$

Remark 1 *i) Since Theorem 1, the zeros of each polynomial* $P_n(t,z)$ *depend on* $t \in \mathbb{R}$ *because its derivatives are no zero. Moreover, in the case of real Toda lattices, this is, when the coefficients* $\alpha_n(t)$ *,* $\lambda_n(t)$ *in (5) are real functions, we have* $\dot{z}_{nk}(t) > 0, k = 1, 2, \ldots, n$*. Then, in this case* $z_{nk}(t)$ *,* $k = 1, 2, \ldots, n$*, are monotonically increasing functions in* $t \in \mathbb{R}$*. For each fixed* n *,* $z_{nk}(t)$ *,* $k = 1, 2, \ldots, n$*, are simple zeros of* $P_n(t,z)$*. Then,* $z_{nk}(t) \neq z_{nk'}(t)$ *for* $k \neq k'$ *,* $k, k' = 1, \ldots, n$*, and, therefore, the curves* $\{z_{nk}(t) : k = 1, 2, \ldots, n\}$ *have not common points.*

ii) Let $J(t)$ *be a bounded operator. It is a consequence of Theorem 2 that* $\|J(t)\|$ *is independent on* $t \in \mathbb{R}$*. Then, for each* $n \in \mathbb{N}$*,*

$$|z_{nk}(t)| \leq \|J(t)\| \leq M\,, \quad k = 1, \ldots, n\,, \, n \in \mathbb{N}\,.$$

From this and i) we deduce

$$\lim_{t \to \infty} z_{nk}(t) = m_k \in \mathbb{R}\,, \quad k = 1, \ldots, n\,,$$

that is, each curve $z_{nk}(t)$ *,* $t \in \mathbb{R}$*, has an asymptotic line* $z = m_k$ *in the* $(t,z)-$*plane.*

iii) When the entries of $J(t)$ *are no real functions, then we don't know the multiplicity of* $z_{nk}(t)$ *as a zero of* $P_n(t,z)$*. Therefore, in this complex case it is possible that i) and ii) are not true.*

2.2 Proof of Theorem 2

We define the antilinear operator $\mathcal{C}$ such that $\mathcal{C}e_i = e_i$ for each vector e_i , $i = 0, 1, \ldots$, in the canonical base. Thus, for any $x \in \ell^2$ we have

$$x = \sum_{i \geq 0} x_i e_i\,, \quad \mathcal{C}x = \sum_{i \geq 0} \overline{x_i} e_i\,.$$

In [9], for the study of symmetric complex operators, some antilinear operators were used. In our case, we have the following auxiliar result for $\mathcal{C}$, which justifies the definition of transpose operator (see [9, pag. 2]). We recall the we call in the same way an operator and it matricial representation.

Lemma 1 *a) Let* A *be a linear operator and let* A^* *be the adjoint operator of* A*. Then, the matricial representation of* $\mathcal{C}A^*\mathcal{C}$ *is* A^T*, this is,* $A^T = \mathcal{C}A^*\mathcal{C}$*.*

b) $J(t)$ *is a symmetric complex operator, this is,* $J(t) = \mathcal{C}J(t)^*\mathcal{C}$ *for each* $t \in \mathbb{R}$*.*

c) $K(t)$ *is an antisymmetric operator, this is,* $K(t) = -\mathcal{C}K(t)^*\mathcal{C}$ *for each* $t \in \mathbb{R}$*.*

Proof.- Given a linear operator B, it is obvious that $\mathcal{C}B\mathcal{C}$ is also a linear operator. So, it is sufficient to prove the enunciated equalities for each basic vector e_i. In a), for $A = (a_{ks})_{k,s=0}^{\infty}$, the column i A^* is given by

$$A^* e_i = \sum_{k \geq 0} \overline{a_{ik}} e_k,$$

therefore,

$$\mathcal{C}A^*\mathcal{C}e_i = \sum_{k\geq 0} a_{ik}e_k\,,$$

which is the column i of the transpose matrix A^T. For proving *b)* and *c)*, it is sufficient to take in account the following expressions,

$$\left.\begin{array}{rcl} J(t)e_i &=& \lambda_{i+1}(t)e_{i-1}+\alpha_{i+1}(t)e_i+\lambda_{i+2}(t)e_{i+1} \\ 2K(t)e_i &=& -\lambda_{i+1}(t)e_{i-1}+\lambda_{i+2}(t)e_{i+1} \end{array}\right\}\,,\quad i=0,1,\dots\,,$$

where we understand $e_{-1}=0$. In other words, *b)* and *c)* can be obtained directly as a consequence of the structure of matrices $J(t)$ and $K(t)$. $\square$

Now, we consider the following matricial initial value problem:

$$\left.\begin{array}{rcl} \dot{Q}(t) &=& Q(t)K(t) \\ Q(0) &=& I \end{array}\right\} \tag{15}$$

Under restrictions of Theorem 2, the operator $K(t)$ given in (3) is bounded. Hence, we assume this condition in the rest of the section. Moreover, assuming continuous solutions for the Toda lattice, the operator $K(t)$ is a continuous function on $t\in\mathbb{R}$. It is known that we can consider different kinds of continuity for a operator-value function $t\mapsto A(t)$. In our case, the function $t\mapsto K(t)$ of a real variable is continuous in norm (see [10, pag. 152]). Therefore, the existence of a solution $Q(t)$ of (15) can be guarantied (see [13, pag. 123]).

We have the following auxiliar result:

Lemma 2 *Let $Q(t)$ be a solution of (15). Then*

$$Q(t)Q(t)^T = Q(t)^TQ(t) = I\,, \tag{16}$$

this is, $Q(t)$ is an invertible matrix and $Q(t)^{-1}=Q(t)^T$

Proof.- Transposing in (15), since Lemma 1 we arrive to

$$\left.\begin{array}{rcl} \dot{Q}(t)^T &=& -K(t)Q(t)^T \\ Q(0)^T &=& I \end{array}\right\} \tag{17}$$

In other words, $Q(t)^T$ is a solution of differential equation $R(t)=-K(t)R(t)$, verifying the same initial condition given in (15).

1. Firstly, we show $Q(t)Q(t)^T=I$. Using (15) and (17) we obtain

$$\frac{d}{dt}\left(Q(t)Q(t)^T\right) = \dot{Q}(t)Q(t)^T + Q(t)\dot{Q}(t)^T = 0\,,$$

then $Q(t)Q(t)^T$ is independent on $t\in\mathbb{R}$. From this fact and $Q(0)=I$ we deduce $Q(0)Q(0)^T=I$ and the first part of (16) is proved.

2. Following [13, pag. 123-124] we can write

$$Q^T(t) = I - \int_0^t K(\tau)d\tau + \sum_{n\geq 2}(-1)^n\int_0^t\int_0^{\tau_1}\cdots\int_0^{\tau_{n-1}} K(\tau_1)K(\tau_2)\cdots K(\tau_n)d\tau_n\cdots d\tau_2 d\tau_1\,. \tag{18}$$

Due to the continuity in norm of $K(t)$, the series given in the right hand side of (18) converges in norm. Then,

$$\|Q^T(t)\| \leq e^{t \max_{[0,t]} \|K(\tau)\|}.$$

In a similar way, the series given in the right hand side of

$$R(t) = I + \int_0^t K(\tau)d\tau + \sum_{n\geq 2}(-1)^n \int_t^0 \int_t^{\tau_1} \cdots \int_t^{\tau_{n-1}} K(\tau_1)K(\tau_2)\cdots K(\tau_n)d\tau_n \cdots d\tau_2 d\tau_1$$

converges in norm. Then, this series defines the bounded operator $R(t)$ for each $t \in \mathbb{R}$. From direct computation we obtain

$$Q^T(t)R(t) = R(t)Q^T(t) = I\,, \quad t \in \mathbb{R}\,.$$

Then, because of $Q(t)Q^T(t) = I$ we deduce $R(t) = Q(t)$ and, finally, $Q^T(t)Q(t) = I$.

□

Now, we shall finish the proof of Theorem 2. For this purpose, take a solution $Q(t)$ of (15). Using (2), (15) and (17) we immediately arrive to

$$\frac{d}{dt}\left(Q(t)J(t)Q^T(t)\right) = \dot{Q}(t)J(t)Q(t)^T + Q(t)\dot{J}(t)Q(t)^T + Q(t)J(t)\dot{Q}(t)^T = 0\,.$$

Then, taking into account the initial condition in (15) and (17),

$$Q(t)J(t)Q(t)^T = Q(0)J(0)Q(0)^T = J(0)\,.$$

From this and Lemma 2,

$$J(t) = Q(t)^T J(0)Q(t)\,.$$

This is, $J(0)$ and $J(t)$ are equivalent and we have, as a consequence,

$$\sigma\left(J(t)\right) = \sigma\left(J(0)\right)$$

for each $t \in \mathbb{R}$. So, $\sigma\left(J(t)\right)$ is independent on $t \in \mathbb{R}$, as we wanted to prove. □

3 Obtaining some new solutions of Toda lattice

Consider the following solution of (1):

$$\left.\begin{array}{r} \alpha_n(t) = e^t + n - 1 \\ \lambda_n(t) = \sqrt{(n-1)e^t} \end{array}\right\}\,,\ n \in \mathbb{N}\,. \tag{19}$$

With the notation employed in the above sections we have

$$J(t) = \begin{pmatrix} e^t & \sqrt{e^t} & & \\ \sqrt{e^t} & e^t + 1 & \sqrt{2e^t} & \\ & \sqrt{2e^t} & e^t + 2 & \ddots \\ & & \ddots & \ddots \end{pmatrix}. \tag{20}$$

Since $\sum_{n\geq 1}\frac{1}{\sqrt{ne^t}}$ is a divergent series, the Carleman condition ([12, pag. 59]) indicates that $J(t)$ is a selfadjoint operator. Moreover, it is easy to see that $\det\left(J_n(t)\right)=e^{nt}$, $n\in\mathbb{N}$ and, therefore, $J(t)$ is a positive-definite operator. From both things we can deduce

$$\sigma\left(J_n(t)\right)\subset[0,+\infty)\,.$$

Then, (19) is an example of solution of (1) for which the associated polynomials $\{P_n(t,z)\}$ have all their zeros in $[0,+\infty)$. The dynamic behavior of these zeros was determined in Theorem 1 and Remark 1.

From (19) it is possible to obtain some complex solutions of (1). With this purpose we take $C\in\mathbb{C}\setminus[0,+\infty)$ and we apply the method given in [2]. Here, wee explain and illustrate that method. Let

$$J^{(1)}(t):=\begin{pmatrix} e^t & e^t & & \\ 1 & e^t+1 & 2e^t & \\ & 1 & e^t+2 & \ddots \\ & & \ddots & \ddots \end{pmatrix}$$

be. Due to $P_n(t,C)\neq 0$, we have

$$\det\left(J_n(t)-CI_n\right)=\det\left(J_n^{(1)}(t)-CI_n\right)\neq 0\,.$$

Thus, $J^{(1)}(t)-CI$ admits a formal representation like

$$J^{(1)}(t)-CI=L(t)U(t)\,,$$

([8, Th. 1, pag. 35]), being

$$L(t):=\begin{pmatrix} l_{11}(t) & & & \\ l_{21}(t) & l_{22}(t) & & \\ & l_{32}(t) & l_{33}(t) & \\ & & \ddots & \ddots \end{pmatrix}\,,\; U(t):=\begin{pmatrix} 1 & u_{12}(t) & & & \\ & 1 & u_{23}(t) & & \\ & & 1 & u_{34}(t) & \\ & & & \ddots & \ddots \end{pmatrix}$$

(Despite the entries in both matrices depend on C, by brevity we don't write explicitly this dependence.) In a more precisely way, for each $m\in\mathbb{N}$ we obtain

$$l_{mm}(t)=e^t+m-1-C-\cfrac{(m-1)e^t}{e^t+m-2-C-\cfrac{(m-2)e^t}{\ddots-\cfrac{e^t}{e^t-C}}}\,,\; l_{m+1,m}(t)=1\,,\; u_{m,m+1}(t)=\frac{me^t}{l_{mm}(t)}.$$

The new obtained solution, generated from $J(t)$ and C, is given as $\{\widetilde{\alpha}_n(t)\,,\,\widetilde{\lambda}_{n+1}(t)\}\,,\,n\in\mathbb{N}$, being

$$U(t)L(t):=\begin{pmatrix} \widetilde{\alpha}_1(t)-C & \left(\widetilde{\lambda}_2(t)\right)^2 & & \\ 1 & \widetilde{\alpha}_2(t)-C & \left(\widetilde{\lambda}_3(t)\right)^2 & \\ & 1 & \widetilde{\alpha}_3(t)-C & \ddots \\ & & \ddots & \ddots \end{pmatrix}.$$

In other words, for each $C \in \mathbb{C}\backslash[0,+\infty)$ a new solution of the Toda lattice can be generated from the product $U(t)L(t)$. In this way a sequence of solutions can be obtained iterating this process. In our example, the new solution is

$$\widetilde{\alpha}_1(t) = e^t + \frac{e^t}{e^t - C}\,, \quad \widetilde{\alpha}_2(t) = e^t + 1 - \frac{e^t}{e^t - C} + \frac{2e^t}{e^t + 1 - C - \frac{e^t}{e^t - C}}\,, \cdots$$

$$\widetilde{\lambda}_2(t) = \frac{e^t}{e^t - C}\left(e^t + 1 - C - \frac{e^t}{e^t - C}\right)\,, \quad \widetilde{\lambda}_3(t) = \frac{2e^t}{e^t + 1 - C - \frac{e^t}{e^t-C}}\left(e^t + 2 - C\frac{2e^t}{e^t + 1 - C - \frac{e^t}{e^t-C}}\right)\,, \cdots$$

Because our initial solution $\{\alpha_n(t)\,,\ \lambda_{n+1}(t)\}\,,\ n \in \mathbb{N}\,,$ is a real solution, we know $\sigma(J(t)) = \sigma(J(t_0))$ for any $t, t_0 \in \mathbb{R}$. Moreover,

$$\sigma(\widetilde{J}(t)) \setminus C = \sigma(J(t)) \setminus C$$

(see [5, Prop. 3.6, pag. 225]). Thus

$$\sigma(\widetilde{J}(t)) \setminus C = \sigma(J(t_0)) \setminus C\,. \tag{21}$$

Because $\{\lambda_n(t)\}\,,\ n \in \mathbb{N}\,,$ is not a bounded sequence, we can't apply Theorem 2. However, (21) suggests that Theorem 2 would be extended to a more general situation.

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